Analysis Recap

1. Foundation

1.1 Notations

- $\forall x$, for all x
- $\exists x$, there is min. one x
- $\exists! x$, there is exactly one x
- $\nexists x$, there is no x
- $\wedge$, AND
- $\vee$, OR
- $\neg$, NOT
- $\Rightarrow$, Implication
- $\Leftrightarrow$, equivalence
- $A \cup B$, Union
- $A \cap B$, intersection
- $\emptyset$, empty set

Negation Rules:
- $\neg(A \vee B) : \neg A \wedge \neg B$
- $\neg(A \wedge B) : \neg A \vee \neg B$
- $\neg(\forall x, A(x)) : \exists x, \neg A(x)$
- $\neg(\exists x, A(x)) : \forall x, \neg A(x)$
- $\neg(A \Rightarrow B) : A \wedge \neg B$
- $A \Rightarrow B : \neg B \Rightarrow \neg A$ (contraposition)

1.2 Mappings

- Surjective: $\forall y \in B, \exists x \in A, f(x) = y$, image equals whole codomain
- Injectivity: inputs give different outputs, iff $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
- Bijetivity: injective and surjective, f^{-1} exists

1.3 Sums and Products

- Teleskoping Sums:

$$\sum_1^n (a_k - a_{k-1}) = a_n - a_0$$

$$\sum_m^n (a_k - a_{k+1}) = a_m - a_{n+1}$$

- $\sum_1^n \frac{1}{k(k+1)} = 1 - \frac{1}{1+n}$

Products:

- $\prod_{k=1}^n \frac{a_k}{a_{k-1}} = \frac{a_n}{a_0}$
- $\prod_{k=1}^n \left(1 + \frac{1}{n+k}\right) = 2 - \frac{1}{n+1}$

1.4 Supermum and Infimum

Supermum (least upper bound)

- $s = \sup A$ if: $a \leq s$ for all $a \in A$

Infimum (greatest lower bound)

- $i = \inf A$ if: $a \geq i$ for all $a \in A$

$\Rightarrow \inf A = -\sup -A$, $\inf -A = -\sup A$

1.5 Partial Fraction Decomposition

$$f(x) = \frac{P_n(x)}{Q_m(x)}$$

1. if degree (numerator) $\geq$ degree (denumerator) perform polynomial division.

$$\frac{x^4+2x^3+3x^2+4x+5}{(x-1)^2(x^2+1)}$$

$$= \frac{x^4+2x^3+3x^2+4x+5}{x^4-2x^3+2x^2-2x+1}$$

$$= 1 + \frac{4x^3+x^2+6x+4}{(x-1)^2(x^2+1)}$$

2. Compute zeros of denominator

$(x-1)^2 \Rightarrow 2x$

(x^2+1)

3. Set up partial fractions

$(x-1)^2$: $\frac{A}{x-1} + \frac{B}{(x-1)^2}$

(x^2+1): $\frac{Cx+D}{x^2+1}$

$$\frac{4x^3+x^2+6x+4}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

4. Clear denominators:

$$4x^3+x^2+6x+4 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$$

5. Expand term:

$A(x^3 - x^2 + x - 1)$

$B(x^2+1)$

$C(x^3 - 2x^2 + x) + D(x^2 - 2x + 1)$

$x^3: A + C = 4$

$x^2: -A + B - 2C + D = 1$

$x: A + C - 2D = 6$

$1: -A + B + C + D = 4$

6. Solve System

$A = \frac{5}{2}, B = \frac{15}{2}, C = \frac{3}{2}, D = -1$

$\Rightarrow 1 + \frac{5/2}{x-1} + \frac{15/2}{(x-1)^2} + \frac{3/2 x - 1}{x^2+1}$

1.6 Induction

Purpose: Prove $A(n)$ holds for all $n \geq n_0$

1. Base Case:

   Prove $A(n_0)$

2. Induction hypothesis

   Assume $A(n)$ is true

3. Induction Step.

   Prove $A(n+1)$ using hypothesis

For all $n \geq 1$ $\quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

1. Base Case

$n=1 \qquad \sum_1^1 k = 1, \quad \frac{1(1+1)}{2} = 1 \checkmark$

2. Assume $A(n)$ is true

3. $\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$

$$\sum_{k=1}^{n+1} k = \left(\sum_{k=1}^{n} k\right) + (n+1)$$

$$= \frac{n(n+1)}{2} + (n+1)$$

$$= (n+1)\left(\frac{n}{2} + 1\right) = (n+1)\frac{n+2}{2} \checkmark$$

1.7 Triangular inequality

- $|x+y| \leq |x| + |y|$
- $|x-y| \geq ||x| - |y||$

used in:
- convergence proofs
- Normed spaces
- Error estimation

2 Sequences

2.1 Theorems on sequences

Bounded and Monotone Sequences

Monotonically increasing: $a_{n+1} \geq a_n$

Monotonically decreasing: $a_{n+1} \leq a_n$

Bounded: $m \leq a_n \leq M \ \forall n$

- Monotone sequences converge only if bounded.

Cauchy Sequences

$\forall \varepsilon > 0, \exists N \in \mathbb{N}: \forall n, m \geq N, |a_n - a_m| < \varepsilon$

- a sequence converges if and only if it is a Cauchy sequence

→ let $0 \leq c < 1$

$|a_{n+2} - a_{n+1}| \leq c|a_{n+1} - a_n|$

→ then a_n converges (contractive behavior: differences shrink)

Bolzano-Weierstrass Theorem

Every bounded sequence of real or complex numbers have at least one accumulation point (convergent subsequence)

- if (a_n) is bounded → subsequence (a_{n_k}) such that $a_{n_k} \to a$

Used in: Compactness arguments.

Divergence to Infinity

Sequence diverges to $+\infty$ if

$\forall T \in \mathbb{R}, \exists N \in \mathbb{N}: a_n > T \quad \forall n \geq N$

- $a_n \to -\infty$ if $a_n < T$ eventually for all T

2.2 Limits

Sequences:

- $\lim_{n\to\infty} \frac{1}{n^s} = 0 \quad \forall s > 0$
- $\lim q^n = 0 \quad \forall q \in \mathbb{C}, |q| < 1$
- $\lim \sqrt[n]{a} = 1 \quad \forall a \in \mathbb{R}$
- $\lim_{n\to\infty} \frac{1}{n^k} = 0 \quad \forall k \in \mathbb{N}$
- $\lim \sqrt[n]{n} = 1$
- $\lim n^{\pm\frac{1}{n}} = e^{\pm\frac{1}{n}\ln(n)} = 1$
- $\lim_{n\to\infty} \frac{n^k}{2^n} = 0 \quad \forall k \in \mathbb{N}$

Functions:

- $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$
- $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, $\lim_{x\to\infty} \frac{\sin(x)}{x} = 0$
- $\lim_{x\to\infty} \ln(1+x) = \infty$
- $\lim_{x\to\infty} \frac{\ln(x)}{x^n} = 0, \forall n > 0$
- $\lim_{x\to 0^+} \ln(x) = -\infty$, $\lim_{x\to 0^+} \frac{\ln(x)}{x} = -\infty$
- $\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$
- $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$
- $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln(a)$
- $\lim_{x\to 0^+} \frac{1}{x} = \infty$, $\lim_{x\to 0^-} \frac{1}{x} = -\infty$
- $\lim \frac{n}{\ln(1+n)} = \infty$
- $\lim_{n\to\infty} \frac{\ln(n)}{\sqrt[x]{n}} = 0$
- $\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$
- $\lim_{x\to 0} \frac{x^2}{1-\cos(x)} = 2$
- $a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right), c < 1, a_1 = c$; $a \to \sqrt{c}$

l'Hopital

Use cases: when we get $\frac{0}{0}$, $\frac{\infty}{\infty}$, 0^0, ∞^0, 1^∞, (after rewriting $0\cdot\infty$, $\infty-\infty$)

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)} = \dots \lim_{x\to x_0}\frac{f^n(x)}{g^n(x)}$$ *if differentiable*

- if we have $0\cdot\infty$ or $\infty-\infty$ then we first need to
  - $0\cdot\infty$: $f(x)\cdot g(x) = \frac{f(x)}{\frac{1}{g(x)}}$
  - $\infty-\infty$: $f(x)-g(x) = \frac{1}{\frac{1}{f(x)}} - \frac{1}{\frac{1}{g(x)}} = \frac{\frac{1}{g(x)}-\frac{1}{f(x)}}{\frac{1}{g(x)\cdot f(x)}}$

Other methods

- $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln(x)}$ → we look at $\lim_{x\to 0^+} x\ln(x) \to 0$, → $\lim_{x\to 0^+} x^x = e^0 = 1$
- $\lim_{x\to a}(1+\frac{1}{\odot})^{\odot} = e$ / $\lim_{x\to a}(1+\odot)^{\frac{1}{\odot}} = e$ where $\odot\to 0$ if $x\to a$

  $(1+\text{small term})^{\text{large term}}$ | $\lim_{x\to\infty}(1-\frac{3}{x})^{2x} = \lim_{x\to\infty}\left((1+\frac{1}{-\frac{x}{3}})^{-\frac{x}{3}}\right)^{-2x\cdot\frac{3}{x}} = e^{-6}$

- $\lim_{x\to a}\frac{\sin(\odot)}{\odot} = 1$, $\odot\to 0$ if $x\to a$

3. Series

3.1 Theorem on Series

- $\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|$ if $\sum |a_n|$ converges $\rightarrow$ $\sum a_n$ absolute convergence
- Leibniz - Criteria:
  $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ Converges if a_n is monotonically decreasing
- Quotient Criteria:
  $\left|\frac{a_{n+1}}{a_n}\right| \leq c, \forall n \in \mathbb{N}, 0 \leq c < 1$
  $\sum_{n=1}^{\infty} a_n$ converges absolutely
- "Bedingte" Convergence, Dirichlet Criterion:
  $\sum_{k=1}^{\infty} a_k b_k$ converges if:
  - a_n is montonically decreasing $\rightarrow 0$
  - b_n all partial sums are smaller than $C>0$: $\forall n \in \mathbb{N}: \left|\sum_{k=1}^{\infty} b_k\right| \leq C$

  For oscillating terms
  - Fourier type arguments
- Root Criterea:
  $\sum_{n=1}^{\infty} |a_n|$ converges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = q < 1$
  diverges if " >1

  For powers $a_n = x^n$

3.2 Criterias

- $\sum_{1}^{\infty} \frac{1}{n^k}$ $k>1$ converges, $k \leq 1$ diverges
- $\sum_{0}^{\infty} \frac{n^k}{b^n}$ $|b|>1$ converges absolute
- $\sum_{0}^{\infty} \frac{a^n}{n!}$ converges absolute for all $a \in \mathbb{C}$

Null sequence Criteria: (Necessary condition)

- if a_n is not a zero sequence, $\sum_0^\infty a_n$ will diverge

Majorant Criterion:

- $\sum_{n=0}^{\infty} a_n$ konverges if $a_n \leq b_n \; \forall n$ and $\sum_0^\infty b_n$ converges

Minorant Criteria:

- $\sum_{n=0}^{\infty} a_n$ diverges if $a_n \geq b_n \; \forall n$ and $\sum_0^\infty b_n$ diverges
  - Harmonic Series

Integral Criteria:

- $\sum_{n=p}^{\infty} a_n$ converges $\Longleftrightarrow$ $\int_p^\infty a(x)\,dx$ converges ($f: [p, \infty) \to [0, \infty)$, monotonically decreasing)

Power series

- $f(x) = \sum_{n=1}^{\infty} a_n (x - x_0)^n$
- Convergence radius $R = |x - x_0|$

$$R = \frac{1}{Q} = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$
$$= \frac{1}{L} = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$$

Important Sums

- $\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$ Geometric Series
- S_n: $a_0 \frac{q^{n+1} - 1}{q - 1} = a_0 \frac{1 - q^{n+1}}{1 - q}$ (first n of harmonic series, $q \neq 1$)
- $\sum_1^n k = \frac{n(n+1)}{2}$, $\sum_1^n k^2 = \frac{n(n+1)(2n+1)}{6}$, $\sum_1^n k^3 = \left(\frac{n(n+1)}{2}\right)^2$ Arithmetic Series
- $\sum_{n=1}^{\infty} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$ Telescoping series
- $\sum_0^n x^k = \frac{1 - x^{n+1}}{1 - x}$, $\sum_1^n x^k = \frac{x^{n+1} - x}{x - 1}$
- $\sum_0^n (1 + x^{2^k}) = \frac{1 - x^{2^{n+1}}}{1 - x}$
- $\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$ (Riemann Zeta of 2)
- $\sum_1^n \frac{k}{2^k} = 2 - \frac{n+2}{2^n}$

- $\sum_{1}^{n} \frac{1}{4k^2-1} = \frac{n}{2n+1}$

- $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!} \cdots$ Exponential

- $\sum_{k=0}^{\infty} k \cdot \frac{x^k}{k!} = xe^x = \sum_{k=0}^{\infty} \frac{x^k}{(k-1)!}$

- $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sin(x) = \frac{e^{ix}-e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

- $\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = \sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

- $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = \cos(x) = \frac{e^x+e^{-x}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

- $\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = \cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$

- $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2^{2k}-1)\, 2^{2k} B_{2k} x^{2k-1}}{(2k)!} = \tan(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} \cdots$?

- $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} = \arctan(x)$

- $\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots$

- $\sum_{n=0}^{\infty} \binom{\frac{1}{3}}{n} x^n = \sqrt[3]{1+x}$

4. Functions

$f: D \subset \mathbb{R}^n \to \mathbb{R}$ or $\mathbb{C}$

Emphasis on continuity, differentiability, monotonicity, extrema and convergence

4.1 Continuity and basic Theorems

Definition of Continuity:

- $|x-x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$, $\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D$, $f: D \to \mathbb{R}$

  small changes in input cause small changes in output

Order Perservation via derivatives

- If $f(a) \geq g(a)$ and $f'(x) \geq g'(x)$ $\forall x \in (a,b) \Rightarrow f(x) \geq g(x)$ $\forall x\ [a,b]$

Continuity via Estimation

- $|f(x) - f(y)| \leq |g(x) - g(y)|$, for all $x, y \in D$, and g continuous $\Rightarrow$ f also continuous

Lipschitz - Continuity

- $|f(x) - f(y)| \leq c|x-y|$, $c \in \mathbb{R}$, $\forall x, y \in D$ Lipschitz continuity $\Rightarrow$ continuity

Continuity via Sequences

- f continuous at x_0 iff $\forall a_n \subset D$ with $a_n \to x_0$, $f(a_n) \to f(x_0)$

Continuity of Piecewise Functions

$f(x) = \begin{cases} f_1(x) & x < p \\ a & x = p \\ f_2(x) & x > p \end{cases}$ continuous at p iff $\lim_{x \to p^-} f_1(x) \overset{!}{=} \lim_{x \to p^+} f_2(x) \overset{!}{=} a$

Continuity in Higher Dimensions

$\lim_{x \to x_0} f(x)$ exists iff it is unique

- $\mathbb{R}^2$: use polar coordinates: if angle φ remains in the limit $\to$ limit doesn't exist
- $\mathbb{R}^{2+}$: sufficient to show two different paths give different limit.

Intermediate Value Theorem

if f is continoues, $a < b$, $a, b \in D$

if $f(a) < f(b)$, $y \in [f(a), f(b)]$, $x \in [a,b]$

$\to f(x)=y$. The image of the function is also continoues.

Extremum Theorem

$f([a,b])$ has a minimum and maximum.

Monotomicity via derivatives

- $f'(x)>0\ \forall x \Rightarrow$ strictly increasing
- $f'(x)\geq 0\ \forall x \Rightarrow$ monotnically increasing
- $f'(x)<0\ \forall x \Rightarrow$ strictly decreasing
- $f'(x)\leq 0\ \forall x \Rightarrow$ monotonically decreasing

Injectivity and Monotonicity

- f is injective iff it is strictly monotone

iff continuous and strictly monotone
$\Rightarrow f$ is bijective
$\Rightarrow f^{-1}$ is continuous

- Surjective: $\forall y \in B, \exists x \in A, f(x)=y$, image equals whole codomain
- Injectivity: inputs give different outputs, iff $f(x_1)=f(x_2) \Rightarrow x_1=x_2$
- Bijetivity: injective and surjective, f^{-1} exists

Mean Value Theorem

$$\frac{f(b)-f(a)}{b-a}=f'(c), \quad c\in\,]a,b[$$

f: continuous and diffable

Lipschitz-continueous with derivatives

- if f' continueous

$|f(x)-f(y)|\leq M|x-y|$, $x,y\in[a,b]$ with $M=\max|f'(x)|$

Minimum and Maximum of a function

$f'(x_0)=0$
- if $f''(x_0)<0 \to$ Maximum
- if $f''(x_0)>0 \to$ Minimum
- if $f''(x_0)=0 \to$ Saddlepoint

Divergence of Functions

if $\lim_{x\to\infty} f(x)=+\infty \to \forall M>0, \exists R>0, x>R \Rightarrow f(x)>M$

Taylor Series and Std Expansions

- $e^x=\sum_0^\infty \frac{x^k}{k!}$, - $(1+x)^a=1+ax+\frac{a(a-1)}{2}x^2+\cdots$, $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}\cdots$

5 Function Sequences

5.1 Types of Convergence

Pointwise Convergence:

$\lim_{x \to \infty} f_n(x) = f(x), \forall x \in D$

$(\forall x \in D, \forall \varepsilon > 0, \exists N = N(x, \varepsilon) : |f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N)$

For each fixed x, the sequence of numbers $f_n(x_i)$ converges. Different x's need different N's

Example: $f_n(x) = x^n$ on $D = [0,1]$. Pointwise limit:

$$f(x) \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

Uniform Convergence

$\forall \varepsilon > 0 \ \exists N = N(\varepsilon) : \forall n \geq N, \forall x \in D : |f_n(x) - f(x)| < \varepsilon$

N does not depend on x.

- Uniform limit of continous funtions is continous
- Uniform convergence allows interchange of limit and integral
- Uniform convergence is stronger than pointwise convergence

Define $\|f_n - f\|_\infty = \sup_{x \in D} |f_n(x) - f(x)|$

uniform convergence $\Leftrightarrow$ $\|f_n - f\|_\infty \to 0$

Uniform Cauchy Criterion

$\forall \varepsilon > 0 \ \exists N : \forall n, m \geq N, \forall x \in D : |f_n(x) - f_m(x)| < \varepsilon$

f_n is uniformly cauchy $\Leftrightarrow$ it converges uniformly to some bounded f

Easier to estimate pairwise differences than find the limit explicitly

Normal Convergence and Weierstrass M-test

$\sum_n g_n(x)$ $\exists M_n \geq 0$ $\sup_{x\in D} |g_n(x)| \leq M_n$ and $\sum_n M_n < \infty$ then

$\sum_n g_n$ converges normally (uniform absolute convergence)

Normal convergence $\Rightarrow$ uniform convergence $\Rightarrow$ sum is continous if each g_n is continous

M-test: $|x| \leq R$

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ satisfies $\sup_{|x|\leq R} \left|\frac{x^n}{n!}\right| = \frac{R^n}{n!}$ and $\sum \frac{R^n}{n!} = e^R < \infty \rightarrow$ normally convergent on $[-R, R]$

5.2 Practical Recipes & tests Uniform Conv.

1. Find Pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$
2. Try sup-norm: compute $s_n : \sup_{x\in D} |f_n(x) - f(x)|$. if $s_n \to 0$ $\rightarrow$ uniform conv.
3. If direct sup hard try:
   - Weierstrass M-test
   - Estimate by a sequence $b_n \to 0$ with $|f_n(x) - f(x)| \leq b_n$ $\forall x$
   - Dini's theorem: if D compact, f_n continuous, f_n monotone in n and $f_n \to f$ pointwise with continous f, then convergence uniform
4. Negative tests:
   - If limit f is discontinous but each f_n is continous
   - Exhibit an $\varepsilon > 0$ such that for every N there exists $n \geq N$ and $x \in D$ with $|f_n(x) - f(x)| \geq \varepsilon$

$f_n(x) = \frac{x}{n}$, $x \in [0,1]$

pointwise limit: $f_n(x) \to 0$

Error: $|f_n(x)| \leq \frac{1}{n}$

$\sup_{x\in[0,1]} |f_n(x)| = \frac{1}{n} \to 0$

$\rightarrow$ Uniform convergence

$f_n(x) = \frac{x}{n+x^2}$, $x \in \mathbb{R}$

Pointwise limit: $f_n(x) \to 0$

Estimate: $|f_n(x)| = \frac{|x|}{n+x^2} \leq \frac{|x|}{n}$

maximum at $x = \sqrt{n}$ $\quad \frac{x}{n+x^2} \leq \frac{1}{2\sqrt{2}}$

$\sup_x |f_n(x)| \leq \frac{1}{2\sqrt{n}} \to 0$

$\rightarrow$ uniform convergence

5.3 Interchanging limit operations

limit and Continuity / Integral

- if f_n are continuous and $f_n \to f$ uniformly, then f is continuous
- If $f_n \to f$ uniformally on $[a,b]$ then:

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n = \int_a^b f$$

Limit and Derivative

1. There exists $x_0 \in \Omega$ such that $f_n(x_0)$ converges
2. f_n' converge uniformly on Ω to a function g.

$\to$

- f_n converges uniformly on Ω to a C^1-function f
- $f'(x) = g(x) \quad \forall x \in \Omega$

6 Derivatives

6.1 Foundation

Linear approximation (tangent line)

$h(x) = f(x_0) + f'(x_0)(x - x_0)$

Definition:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

derivative = slope of tangent = instantaneous rate of change

Differentiable $\Rightarrow$ Continuous

6.2 Rules

Sum: $(f+g)' = f' + g'$

Product: $(fg)' = f'g + fg'$

Quotient: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Chain: $(f(g(x)))' = f'(g(x)) \cdot g'(x)$

Inverse: $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$ | Change of variables $u = f(x)$ $\quad \frac{dx}{du} = \frac{1}{f'(x)}$

Higher derivatives: $(fg)^k = \sum_{j=0}^{k} \binom{k}{j} f^{(j)} g^{(k-j)}$ (Generalized Product rule)

Integrals: $\frac{d}{dx} \int_{\varphi(x)}^{\psi(x)} f(t)\, dt = f(\psi(x))\psi'(x) - f(\varphi(x))\varphi'(x)$

6.3 Frequent Derivatives

- $(e^x)' = e^x$
- $(a^x)' = \ln(a)\, a^x$
- $\ln(x)' = \frac{1}{x}$
- $(x^a)' = a x^{a-1}$

- $\sin(x)' = \cos(x)$
- $\cos(x)' = -\sin(x)$
- $\tan(x)' = \frac{1}{\cos^2(x)}$
- $\arcsin(x)' = \frac{1}{\sqrt{1-x^2}}$

$y = \arcsin x$, $\sin y = x$, $y' = \frac{1}{\cos y} \Leftarrow \cos y \cdot y' = 1$

$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$

$\arcsin(x)' = \frac{1}{\sqrt{1-x^2}} = \frac{1}{\sin'(\arcsin(y))}$

Inverse rule

- $\arctan(x)' = \frac{1}{1+x^2}$
- $\sinh(x)' = \cosh(x)$
- $\cosh(x)' = \sinh(x)$
- $\tanh(x)' = 1 - \tanh^2(x)$

6.4 Newton's Method

Approximate Zeros of f

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$f(x) = x^2 - 2 \rightarrow f'(x) = 2x$, $x_{n+1} = x_n - \frac{x_n^2-2}{2x_n} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$

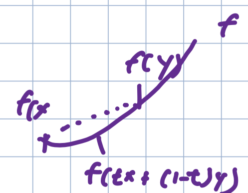
$x_0 = 1$ $x_n = 1.5, 1.4167, 1.4142 \approx \sqrt{2}$

Replace function locally by tangent line and compute where it hits x-axis

6.5 Convex Functions

Definition:

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y)$$



- if $f \in C^2$, $f''(x) \geq 0 \Rightarrow f$ convex

f' is increasing, tangent line is below graph, $f(x) \geq f(x_0) + f'(x_0)(x - x_0)$

- $x < y < z$

$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(x)}{z-x} \leq \frac{f(z)-f(y)}{z-y}$$

Secant Slopes increase

6.6 Taylor Polynomials

$f \in C^k$

Local Approximations

$$T_k f(x; x_0) = \sum_{n=0}^{k} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$ Best approximation of degree k

Remainder (Lagrange Form)

$$f(x) = T_k f(x; x_0) + \frac{f^{(k+1)}(c)}{(k+1)!}(x-x_0)^{k+1} \qquad x \le c \le x_0$$

Error control

Small-o remainder form

$$f(x) = T_k f(x; x_0) + (x-x_0)^k r(x) \quad \text{with } r(x) \to 0$$

Integral Form of Remainder

$$f(x) = T_k f(x; x_0) + \frac{1}{k!}\int_{x_0}^{x} f^{(k+1)}(t)(x-t)^k dt$$ bounding errors providing inequalities

Product of Taylor Series ($f(x) = g(x) \cdot h(x)$)

If $g(x) = \sum a_k x^k$, $h(x) = \sum b_k x^k$

$$\to f(x) = g(x) h(x) = \sum_{n=0}^{\infty} \left(\sum_{k+l=n} a_k b_l \right) x^n$$

Cauchy Product

7 Ordinary Differential Equations (ODEs)

7.1 General Strategy

linear ODE:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} \dots a_0 y = q(x)$$

1. Solve the homogeneous equation ($q(x) = 0$)
2. Find one particular solution of the inhomogeneous equation
3. Use initial conditions to determine constants

The solution space of linear ODE is affine: general solution: $y_h + y_p$

1. Homogeneous Solution

1.: Characteristic Polynomial

$$chp(\lambda) = a_n \lambda^n + \dots a_0 = 0$$

solve for λ

2: Build solution

a) Real root λ (multiplicity k)

$$y_t(x) = x^t e^{\lambda x}, \quad t = 0, \dots k-1$$

$y'' - 2y' + y = 0 \Rightarrow (\lambda - 1)^2 = 0$

$y_h = C_1 e^x + C_2 x e^x$

b) Real roots $\pm a$

often as $y = C_1 \cosh(ax) + C_2 \sinh(ax)$

$\Leftrightarrow e^{ax}, e^{-ax}$

c) Complex roots $a \pm bi$

$$y = e^{ax}(C_1 \cos(bx) + C_2 \sin(bx))$$

$y'' + y = 0 \Rightarrow \lambda = \pm i$

$y = C_1 \cos x + C_2 \sin x$

3.: Combine solutions

$$y_h(x) = \sum C_i y_i(x)$$

2. Particular Solution (Inhomogeneous Part)

Solve: $a_n y^{(n)} + \cdots + a_0 y = q(x)$

Shape: $q(x) = (b_0 + b_1 x \cdots + b_m x^m) e^{\mu x}$

Ansatz: $y_p(x) = x^k (C_0 + C_1 x + \cdots C_m x^m) e^{\mu x}$

k = multiplicity of μ as root of characteristic polynomial

3. Initial Conditions

Order n ODE $\rightarrow$ need n initial values

Solve systems for C.

Examples:

7.2 Matrix Exponential

$$\frac{dF}{dt} = AF(t)$$

Solution: $F(t) = e^{At} C$

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

$F'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} F(t)$, $F(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} C$

8 Differential Calculus in $\mathbb{R}^n$

8.1 Core Concepts

Partial Derivatives:

$$\frac{\partial f}{\partial x_i}(a) = \lim_{h \to 0} \frac{f(a_1, \cdots a_i + h, \cdots a_n) - f(a)}{h}$$

Measures sensitivity of f along coordinate axes.

- Only looks in one direction

$f(x,y) = x^2 y$, $\frac{\partial f}{\partial x} = 2xy$, $\frac{\partial f}{\partial y} = x^2$

Directional Derivative

$$D_v f(x_0) = \lim_{h \to 0} \frac{f(x_0 + hv) - f(x_0)}{h}$$

in any direction

Total Differentiability

$$f(x) \approx f(x_0) + A(x - x_0)$$

$$\lim_{x \to x_0} \frac{\| f(x) - f(x_0) - A(x - x_0) \|}{\| x - x_0 \|} = 0$$

C^1 => totally differentiable => directionally differentiable => partially differentiable => continuous

Jacobian Matrix

$$J_f(x) = df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix}$$

Matrix of first-order partial derivatives

Gradient

$$\nabla f(x) = \begin{pmatrix} \partial f / \partial x_1 \\ \vdots \\ \partial f / \partial x_n \end{pmatrix}$$

Direction of steepest ascent

$D_v f(x) = \nabla f(x) \cdot v$

1-Forms $f(x): \mathbb{R}^n \to \mathbb{R}$

$f(x) = \sum_{i=1}^{n} f_i(x)\, dx_i$

8.2 Function Classes and Theorems

Class C^1

- Partial derivative exist and are continuous

Class C^2

- 2nd order pd exist and continuous

Schwarz's Theorem

If $f \in C^2$ then

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

Inverse Function Theorem

Iff $f \in C^1$, $df(x_0)$ invertible

$\Rightarrow$

f is locally invertible

$(df(x_0))^{-1} = dg(f(x_0))$

Diffeomorphism

A bijective C^1 map with C^1 inverse

Smooth change of coordinates

Showing diffeomorphism:

1. show C^1

2. show $\det d\Phi(x) \neq 0$

3. show global bijection or compute inverse explicitly.

Implicit Function Theorem

Problem: Solve $f(x,y)=0$ for $y=h(x)$

$\det(d_y f) \neq 0$

Result: Local solution exists
$h \in C^1$
Derivative: $dh = -(d_y f)^{-1} d_x f$

Existance of Extrema

if - $\Omega \subset \mathbb{R}^n$ compact
- f continous

$\Rightarrow$ f attains min and max

8.3 Differential Rules

- Sum: $d(f+g)(x_0) = df(x_0) + dg(x_0)$
- Product ($\Omega \to \mathbb{R}$): $d(f \cdot g)(x_0) = f(x)\, dg(x_0) + df(x_0)\, g(x_0)$
- Quotient: $d\left(\frac{f}{g}\right)(x_0) = \dfrac{g(x_0)\, df(x_0) - f(x_0)\, dg(x_0)}{g(x_0)^2}$

Chain Rule - 1D:

$g: \Omega \to \mathbb{R}$, $f: \mathbb{R} \to \mathbb{R}$

$$d(f \circ g)(x_0) = f'(g(x_0))\, dg(x_0)$$

Chain Rule - General:

$g: \mathbb{R}^n \to \mathbb{R}^m$, $f: \mathbb{R}^m \to \mathbb{R}^o$

$$d(f \circ g)(x_0) = df(g(x_0)) \circ dg(x_0)$$

Jacobian:

$$J_{f \circ g}(x_0) = J_f(g(x_0)) \cdot J_g(x_0)$$

8.4 Taylor expansion in R^n

2nd Order (Two-Variables)

$\Delta x = x - x_0$, $\Delta y = y - y_0$

$$f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$$
$$+\frac{1}{2}\left(f_{xx}(x_0,y_0)(\Delta x)^2 + 2f_{xy}(x_0,y_0)\Delta x\Delta y + f_{yy}(x_0,y_0)(\Delta y)^2\right)$$
$$+ R_2(\Delta x, \Delta y)$$

Multivariate

$$T_N f(x,a) = \sum_{k=0}^{N} \frac{1}{k!}\left(\Delta_1 \partial_{x_1} + \dots + \Delta_n \partial_{x_n}\right)^k f(x)\Big|_{x=a}$$

8.5 Practical Recipes

Check Differentiability

- Typically combinations of differentiable functions → but issue usually around origin

1. Identify problematic points
2. Check Continuity at the point
   - use polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$ and study limit as $r \to 0$
   - test sequences approaching along different curves
3. Compute partial derivatives at the point
   - partial derivatives
   - For origin, compute $\partial_x f(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$
4. Candidate differentials Jacobian A
5. Test Fréchet differentiability

$$\frac{\| f(x) - f(x_0) - A(x - x_0)\|}{\|x - x_0\|}$$
$x \to x_0$

If limit $\neq 0$ → not differentiable

1. Or show partials exist in a neighborhood and are continous

Recipe to show f is C^1

1. partial derivatives exist everywhere in the domain
2. Show each partial is continous everywhere
   - show combination of smooth functions.
3. $f \in C^1$

9 Extremas

- Maximum or Minimum
- Globally or Locally
- Constraints or no

9.1 Unconstrained Optimization

$f: \Omega \subset \mathbb{R}^n \to \mathbb{R}$

Local Minimum: $\exists \varepsilon > 0$, $f(x_0) \leq f(x)$ $\forall x \in B_\varepsilon(x_0)$

Local maximum: $f(x_0) \geq f(x)$, $\forall x \in B_\varepsilon(x_0)$

Global: $x \in \Omega$

1. One-Dimensional

$f: [a,b] \to \mathbb{R}$ or open interval

1. Boundary points

   $x = a$, $x = b$

2. Critical points:

   $f'(x_0) = 0$

   or where f' does not exist (cusps, corners)

3. Classify Extremas:

   $f'(x_0) = 0$

   Minimum: $f''(x_0) > 0$

   Max : $f''(x_0) < 0$

   Inconclusive: $f''(x_0) = 0$

4. Global?

   - Evaluate candidates f
   - Compare value

2. Multivariable Functions

$f: \Omega \subset \mathbb{R}^n \to \mathbb{R}$

1. Candidates

$$\nabla f(x_0) = 0$$

2. Hessian

$$H_f(x_0) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

3. Classification:

local min: H_f positive definite or EVal > 0

local max: H_f negative definite or EVal < 0

Saddlepoint: indefinite or mixed signs of EVal

Inconclusive: semi definite

9.2 Constrained Optimization

$S = \{ x \in \mathbb{R}^n \mid g(x) = 0 \}$

At extremum on S, gradient of f must be orthogonal to constraint surface.

∇f in the span of ∇g

1. Single constraint

- $f: \mathbb{R}^n \to \mathbb{R}$, $g: \mathbb{R}^n \to \mathbb{R}$
- Constraint: $g(x) = 0$
- Regular point: $\nabla g(x_0) \neq 0$

x_0 is local extremum of f on S then:

$$\nabla f(x_0) = \lambda \nabla g(x_0)$$

2. Multiple constraints

$$\nabla f(x_0) = \sum_{i=1}^{l} \lambda_i \nabla g_i(x_0)$$

Lagrange Function

$$L(x,\lambda) = f(x) + \sum_{i=1}^{c} \lambda g_i(x)$$

Candidate:

- $\nabla_x L(x,\lambda) = 0$ and $g(x) = 0$

Steps:

1. Understand constraint set
2. check Compactness
3. Compute Gradients
4. Solve Lagrange System
5. List Candidate Points
6. Global: Evaluate f
7. Local: use Hessian of Lagrangian

Example:

$$f(x,y,z) = 4y - 2z$$

Constraints: $\varphi_1(x,y,z) = x^2 + y^2 - 1 = 0$

$\varphi_2(x,y,z) = 2x - y - z - 2 = 0$

1. Intersection of Cylinder and plane
2. Closed and Bounded
3. $\nabla f = \begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix} \quad \nabla \varphi_1 = \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix} \quad \nabla \varphi_2 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$
4. Lagrange System

$$\nabla f = \lambda \nabla \varphi_1 + \lambda_2 \nabla \varphi_2 \qquad \left| \begin{array}{l} 0 = 2\lambda_1 x + 2\lambda_2 \\ 4 = 2\lambda_1 y - \lambda_1 \\ 2 = \lambda_2 \end{array} \right. \qquad \begin{array}{l} x^2 + y^2 = 1 \\ 2x - y - z = 2 \end{array}$$

$\lambda_2 = 2, \lambda_1 = \pm\sqrt{13}, \quad x = \mp \frac{2}{\sqrt{13}}, y = \pm \frac{3}{\sqrt{13}}, z = \frac{-7}{\pm\sqrt{13}} - 2$

$$P_1 = \left(\frac{-2}{\sqrt{13}}, \frac{3}{\sqrt{13}}, \frac{-7}{\sqrt{13}} - 2\right)$$

$$P_2 = \left(\frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}}, \frac{7}{\sqrt{13}} - 2\right)$$

5. Evaluate

$$f(P_1) > f(P_2)$$

Switch to .md file

10. Integration

10.1 Definitions and Foundations

Step Functions

$$\int_a^b s(t)\,dt = \sum_{i=0}^{k-1} \sigma_i (x_{i+1} - x_i)$$

$s(t) = 2$ on $[0,1)$, $s(t) = -1$ on $[1,3)$, $s(t) = 3$ on $[3,4]$

$$\int_0^4 s(t)\,dt = 2\cdot(1-0) + (-1)\cdot(3-1) + 3\cdot(4-3) = 2-2+3 = 3$$

Fundamental Theorem of Calculus

differentiation and integration are inverse operations

$$F(x) = \int_a^x g(t)\,dt \Rightarrow F'(x) = g(x)$$

evaluation

$$\int_a^b g(x)\,dx = G(b) - G(a)$$

$$F(x) = \int_0^x (t^3 + \cos t)\,dt$$

$$F'(x) = x^3 + \cos x,\; F(0) = 0$$

$$\int_0^\pi (t^3 + \cos t)\,dt = \left|\frac{t^4}{4} + \sin(t)\right|_0^\pi = \frac{\pi^4}{4}$$

Riemann sums

integral is the limit of sums of thin rectangle areas

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=0}^{n-1} f\left(a + k\cdot\frac{b-a}{n}\right)$$

$$\lim_{n\to\infty} \frac{2}{n} \sum_{k}^{n} \left(1+\frac{2k}{n}\right)^2 \to a=0, b=2,\ x_k=\frac{2k}{n},\ \Delta x=\frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \int_0^2 (1+x)^2\,dx = \left|\frac{(1+x)^3}{3}\right|_0^2 = 9-\frac{1}{3}$$

Limit and Integral

if $f_n \to f$ uniformly on $[a,b]$ then

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx$$

$f_n(x) = x + \frac{\sin(nx)}{n}$ on $[0,1]$, Pointwise $f_n(x) \to x$

$$\|f_n - x\|_\infty = \sup_{x\in[0,1]} \left|\frac{\sin(nx)}{n}\right| < \frac{1}{n} \to 0$$

Convergence is uniform:

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 \lim f_n(x)\,dx = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

Improper Integrals

Infinite interval:

$$\int_a^\infty g(t)\,dt = \lim_{R\to\infty} \int_a^R g(t)\,dt$$

Endpoint singularity

$$\int_a^b g(t)\,dt = \lim_{x\to a} \int_x^b g(t)\,dt$$ g is not defined at a

Two sided interval

$$\int_{-\infty}^{\infty} g(t)\,dt = \int_{-\infty}^{0} g(t)\,dt + \int_0^\infty g(t)\,dt$$

both side must converge.

$\int_1^\infty \frac{1}{x^p}\,dx$ for $p \neq 1$

$$\int_1^R x^{-p}dx = \frac{x^{1-p}}{-p}\Big|_1^R = \frac{R^{1-p}-1}{1-p}$$

$R \to \infty$ converges if $p>1$ with value $\frac{1}{1-p}$

diverges for $p \leq 1$

Singularity

$$\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{\varepsilon \downarrow 0} \int_\varepsilon^1 x^{-\frac{1}{2}}dx = 2 \cdot x^{\frac{1}{2}}\Big|_\varepsilon^1 = 2$$

Inequalities with integrals

For $x \in [0,1]$

$$1-x \leq e^{-x} \leq 1$$

$\Rightarrow$

$$\int_0^1 (1-x)dx \leq \int_0^1 e^{-x}dx \leq \int_0^1 1\,dx$$

$$\frac{1}{2} \leq 1-e^{-1} \leq 1$$

Abosolute Value Estimate

$$\left|\int_0^a \sin(t^2)dt\right|$$

since $|\sin(t^2)| \leq 1$

$$\left|\int_0^a \sin(t^2)dt\right| \leq \int_0^a |\sin(t^2)|dt \leq \int_0^a 1\,dt = a$$

Line Integral

Scalar field: $\gamma: [a,b] \to \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}$

$$\int_\gamma f\,ds = \int_a^b f(\gamma(t))\,\|\dot{\gamma}(t)\|_2\,dt$$

Vector field: $v: \mathbb{R}^n \to \mathbb{R}^n$

$$\int_\gamma v \cdot dr = \int_a^b v(\gamma(t)) \cdot \gamma'(t)\,dt$$

$\gamma(t) = (t, t^2),\ t \in [0,1]$

$\Rightarrow \gamma'(t) = (1, 2t),\ \|\gamma'(t)\| = \sqrt{1+4t^2}$

Scalar: $f(x,y) = x \rightarrow f(\gamma(t)) = t$

$$\int_\gamma f\,ds = \int_0^1 f(\gamma(t)) \cdot \|\gamma'(t)\|_2\,dt = \int_0^1 t \cdot \sqrt{1+4t^2}\,dt, \quad u = 1+4t^2 \quad \begin{array}{l} u(0) = 1 \\ u(1) = 1+4 = 5 \end{array}$$

$du = 8t \cdot dt$

$$\Rightarrow \int_1^5 \cancel{t}\, u^{\frac{1}{2}} \frac{du}{8\cancel{t}} = \frac{1}{8}\int_1^5 u^{\frac{1}{2}}\,du = \frac{1}{8} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_1^5 = \frac{1}{12}(5^{3/2} - 1)$$

Vector: $V(x,y) = (y,x) \rightarrow v(\gamma(t)) = (t^2, t)$

$$\int_\gamma v \cdot dr = \int_0^1 v(\gamma(t)) \cdot \gamma'(t)\,dt = \int_0^1 (t^2, t) \cdot (1, 2t)\,dt = \int_0^1 t^2 + 2t^2 = \int_0^1 3t^2\,dt$$

$$= t^3 \Big|_0^1 = 1$$

Conservative Vector Fields

$\nabla \phi = v$

$\oint_\gamma v \cdot dr = 0$

$\rightarrow \int_A^B v \cdot dr = \phi(B) - \phi(A)$

$v(x,y) = (2x, 2y) = \nabla(x^2+y^2)$, $\phi(x,y) = x^2+y^2$, $A = (0,0)$, $B = (1,2)$

$$\int_A^B v \cdot dr = \phi B - \phi A = (1^2 + 2^2) - 0 = 5$$

10.2 Integration Rules

Integration by Parts, Partielle Integration

$$\int u v' = uv - \int v u'$$

$$\int_a^b u(x)\, v'(x)\, dx = \Big| u(x)\, v(x) \Big|_a^b - \int_a^b u'(x)\, v(x)\, dx$$

$$\int_0^1 x \ln(1+x)\, dx$$

$v' = x\, dx,\ u = \ln(1+x) \Rightarrow v = \frac{1}{2}x^2,\ u' = \frac{1}{1+x}\, dx$

$$\Rightarrow \int_0^1 x \ln(1+x)\, dx = \left| \frac{1}{2}x^2 \cdot \ln(1+x) \right|_0^1 - \int_0^1 \frac{x^2}{2} \cdot \frac{1}{1+x}\, dx$$

$$\begin{array}{r} x-1 \\ x+1 \,\big)\, \overline{x^2} \\ x^2+x \\ \hline -x \\ -x-1 \\ \hline +1 \end{array}$$

$$= \frac{1}{2} \cdot \ln(2) - \frac{1}{2}\int_0^1 \frac{x^2}{1+x}\, dx = \frac{1}{2}\ln(2) - \frac{1}{2}\int_0^1 x - 1 + \frac{1}{x+1}$$

$$= \frac{1}{2}\ln(2) - \frac{1}{2}\left[\frac{1}{2}x^2 - x + \ln(x+1)\right]_0^1 = \frac{1}{2}\ln(2) - \frac{1}{2}\left(\frac{1}{2} - 1 + \ln(2) - \ln(1)\right)$$

Substitution Rule

If $u = h(t)$, $du = h'(t)\, dt$

$$\int h'(t) \cdot g(h(t)) = \int g(u)\, du$$

$$\int_{t=a}^{t=b} h'(t) \cdot g(h(t))\, dt = \int_{u=h(a)}^{u=h(b)} g(u)\, du$$

$f'(x)\, g(f(x))$. $(ax+b)^n$, $e^{h(x)}$, $\sin(h(x))$, $\cos(h(x))$

$$\int_0^1 \frac{2x}{(1+x^2)^2}\,dx$$

$u = 1+x^2$, $du = 2x\,dx$, $u: 1 \to 2$

$$\int_1^2 \frac{1}{u^2}\,du = \int_1^2 u^{-2}\,du = -u^{-1}\Big|_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$$

linear Change of Variable

$u = at + b$

$$\int f(at+b)\,dt = \frac{1}{a}\int f(u)\,du$$

$$\int_{x_0}^{x_1} f(at+b)\,dt = \frac{1}{a}\int_{ax_0+b}^{ax_1+b} f(u)\,du$$

$\int_0^2 e^{3x-1}\,dx$, $u = 3x-1$, $du = 3dx$, $dx = \frac{du}{3}$, bouds $\to -1 \to 5$

$$\frac{1}{3}\int_{-1}^{5} e^u\,du$$